Polygons inscribed in conics in $\mathbb{CP}_2$.

An algebraic geometric approach.

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Introduction to projective geometry.

Definition

The complex projective plane $\mathbb{CP}_2$ is the quotient space $\mathbb{C}^3/\sim$, where for any two points $x \sim y$ iff one is a scalar multiple of the other.

Definition

The dual plane $\mathbb{CP}_2^*$ consists of one-dimensional subspaces (lines) of $\mathbb{CP}_2$.

Projective duality associates every point of $\mathbb{CP}_2$ with a corresponding line in $\mathbb{CP}_2^*$.

Definition

A linear transformation $\mathbb{C}^3 \to (\mathbb{C}^3)^*$ induces a projective-linear transformation $\mathbb{CP}_2 \to \mathbb{CP}_2^*$.

Polygons inscribed in conics.

Let P be a polygon with vertices $A_1, \ldots, A_n$.

Definition

Let T_k be a map that takes P to the polygon in the dual plane with vertices $\overline{A_i A_{i+k}}$, $1 \leq i \leq n$. $T_1(P)$ is the dual polygon of P.

Note that T_k is an involution, that is, $T_k \circ T_k$ is the identity.

We let $T_{jk} = T_j \circ T_k$.

Definition

We say that a polygon P is projectively equivalent to a polygon Q in the dual space if there exists a projective transformation f such that $f(P) = Q$. We write $P \sim Q$.

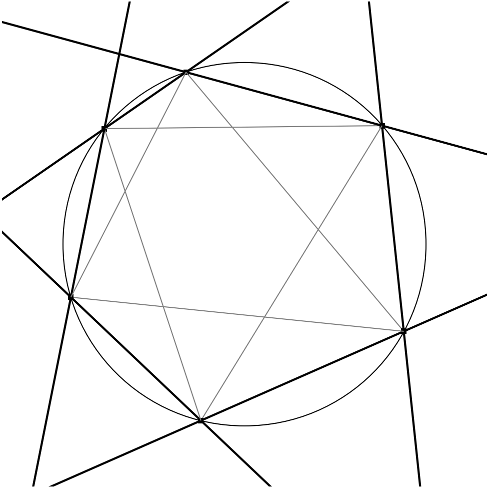
Overview of previous results.

Schwartz and Tabachnikov [ST] found the following theorems through computer experimentation while studying the pentagram map (T_{12}).

Theorem

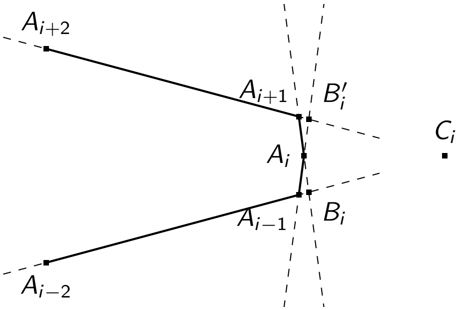
1. *If P is a 6-gon inscribed in a conic, then $P \sim T_2(P)$.*
2. *If P is a 7-gon inscribed in a conic, then $P \sim T_{212}(P)$.*
3. *If P is a 8-gon inscribed in a conic, then $P \sim T_{21212}(P)$.*
4. *If P is an 9-gon inscribed in a conic, then $P \sim T_{1313}(P)$.*
5. *If P is a 12-gon inscribed in a conic, then $P \sim T_{3434343}(P)$.*

Figure: $T_{12}(P)$ when P is a 6-gon.



Schwartz and Tabachnikov were able to prove the first two statements by a method called corner invariants.

For an n-gon with vertices A_i we can consider the following points:
$B_i := \overline{A_{i-2}A_{i-1}} \cap \overline{A_iA_{i+1}}$, $B'_i := \overline{A_{i+2}A_{i-1}} \cap \overline{A_{i-1}A_i}$, and
$C_i := \overline{A_{i-2}A_{i-1}} \cap \overline{A_{i+2}A_{i+1}}$.



Then we have the following cross-ratios for each i:

$$x_i = [A_{i-2}, A_{i-1}, B_i, C_i], y_i = [A_{i+2}, A_{i+1}, B'_i, C_i].$$

This gives us a sequence of invariants $x_1, y_1, x_2, y_2, \ldots$ which we call corner invariants. The sequence is periodic with $x_k = x_{k+n}$, $y_k = y_{k+n}$. It can be shown that corner invariants are invariant under projective transformation and that they define the n-gon uniquely. It turns out we have some interesting relations between the corner invariants of P and $T_1(P)$ [FT09].

Lemma

Let P be an n-gon with vertices A_i. Let x_i, y_i be its corner invariants and let x_i', y_i' be the corner invariants of $T_1(P)$. Then $x_i' = y_i$ and $y_i' = x_{i+1}$.

The proofs of Schwartz and Tabachnikov use these properties of corner invariants to form some geometric arguments that work in the cases $n = 6$ and $n = 7$.

The other cases were proved by brute force computation.

Schwartz and Tabachnikov also discovered a second set of theorems.

Theorem

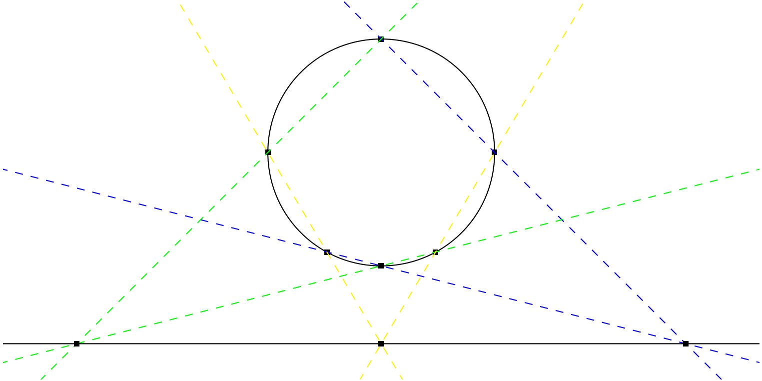
1. *If P is an 8-gon inscribed in a conic, then $T_{13}(P)$ is also inscribed in a conic.*
2. *If P is a 10-gon inscribed in a conic, then $T_{1313}(P)$ is also inscribed in a conic.*
3. *If P is a 12-gon inscribed in a conic, then $T_{131313}(P)$ is also inscribed in a conic.*

They were able to verify the first two statements algebraically, checking several identities symbolically. The third statement however requires symbolic manipulation that Mathematica was not able to handle, and so far it has not been proven.

Some geometrical proofs were found for the first two statements by S. Wang [Wan]. The main tool is Pascal's theorem.

Theorem (Pascal)

Six points A_i lie on a conic if and only if $\overline{A_1A_2} \cap \overline{A_4A_5}$, $\overline{A_2A_3} \cap \overline{A_5A_6}$, $\overline{A_3A_4} \cap \overline{A_6A_1}$ are collinear.



Now let's look at Wang's proof for $n = 8$.

Proof.

Let P have vertices $A_1, \ldots, A_8$. Let $B_i = \overline{A_i A_{i+3}} \cap \overline{A_{i+1} A_{i+4}}$. Apply Pascal's theorem to the six points $A_6, A_1, A_4, A_7, A_2, A_5$ to see that B_1, B_6 and $\overline{A_4 A_7} \cap \overline{A_5 A_6}$ are collinear. This is equivalent to the concurrency of lines $\overline{B_1 B_6}, \overline{B_3 B_4}, \overline{A_5 A_6}$. Thus A_5, A_6 and $\overline{B_6 B_1} \cap \overline{B_3 B_4}$ are collinear since $A_5 = \overline{B_1 B_2} \cap \overline{B_4 B_5}$ and $A_6 = \overline{B_2 B_3} \cap \overline{B_5 B_6}$. So by Pascal's theorem we can conclude that $B_1, \ldots, B_6$ lie on some conic.

By the same argument, $B_2, \ldots, B_7$ lie on a conic. Five points determine a unique conic, so $B_1, \ldots, B_7$ must lie on the unique conic where $B_2, \ldots, B_5$ lie. We can conclude by the same argument that B_8 also lies on this conic, so we are done. □

Remarks

- It is very strange that there seems to be a very clear pattern for both sets of theorems, but they do not hold for higher values of n. The existing proofs do not provide much insight on this, so our goal was to understand why the theorems hold for only certain cases, and whether there is a way to generalize the results.
- We were unsuccessful in finding a new approach to studying projective equivalence of polygons: it seems corner invariants remain the best approach, though they do not generalize easily.
- However, for the second set of theorems we did find algebraic geometric proofs (for $n = 8$ and $n = 10$) that are a little more insightful than the existing proofs.

A different approach

We will now consider a new approach to the following results:

- If P is an 8-gon inscribed in a conic, then $T_{13}(P)$ is also inscribed in a conic.
- If P is an 10-gon inscribed in a conic, then $T_{1313}(P)$ is also inscribed in a conic.

Our approach will consist of a clever application of Max Noether's $AF + BG$ theorem.

Max Noether's $AF + BG$ theorem

The $AF + BG$ theorem describes when a polynomial can be written as a sum of polynomial multiples of two other polynomials.

Theorem

Let C and D be curves in $\mathbb{CP}_2$ with no common components, and let F and G be the homogeneous polynomials that define them. For any homogeneous polynomial H there exist homogeneous polynomials A and B of degrees $\deg H - \deg F$ and $\deg H - \deg G$, respectively, such that $H = AF + BG$ iff Noether's conditions are satisfied at every point in $F \cap G$.

All we need to know about Noether's conditions is that they are satisfied if F and G meet in $(\deg F)(\deg G)$ distinct points, and H passes through these points.

Bézout's theorem

We also need Bézout's theorem.

Theorem

Let C and D be curves in $\mathbb{CP}_2$ of degree m and n, respectively. If they have no common components, then they intersect in at most mn points.

We won't prove it in full detail, but here is a brief sketch.

Proof.

Let f and g be the homogeneous polynomials that define C and D. Take the resultant $R(f,g)$, which is a homogeneous polynomial of degree mn. By the Fundamental Theorem of Algebra it can be split into mn (not necessarily distinct) linear factors. Each of these factors corresponds to an intersection point. □

An interesting corollary of the $AF + BG$ theorem

Corollary

Let C, D_1 and D_2 be curves in $\mathbb{CP}_2$ such that C is of degree c, and D_1, D_2 are of equal degree d and have no common component. Suppose $D_1 \cap D_2$ consists of d^2 points, and $C \cap D_1 \cap D_2$ consists of cd points. Then the $d^2 - cd$ points of $(D_1 \cap D_2) \setminus C$ lie on a curve of degree $d - c$.

Proof.

Let f, g_1, g_2 be the homogeneous polynomials that define C, D_1, D_2, respectively. By Bézout's theorem, $C \cap D_1$ consists of no more than cd points. But $C \cap D_1 \cap D_2$ consists of cd points, so $C \cap D_1$ consists of exactly cd points. Then Noether's conditions are satisfied so we can apply the $AF + BG$ theorem to see that there must exist homogeneous a and b of degrees $d - c$ and $d - d = 0$ such that $g_2 = af + bg_1$. But at any point on $(D_1 \cap D_2) \setminus C$, g_1 and g_2 vanish while f doesn't, so a must vanish. Thus the points of $(D_1 \cap D_2) \setminus C$ lie on a, which has degree $d - c$. □

A proof of Pascal's theorem

This corollary leads to an interesting proof of Pascal's theorem. Recall that the theorem states:

Theorem (Pascal)

Six points A_i lie on a conic if and only if $P = \overline{A_1A_2} \cap \overline{A_4A_5}$, $Q = \overline{A_2A_3} \cap \overline{A_5A_6}$, $R = \overline{A_3A_4} \cap \overline{A_6A_1}$ are collinear.

Proof.

Let C be the conic (a curve of degree two), $D_1 = \overline{A_1A_5} \cup \overline{A_2A_4} \cup \overline{A_3A_6}$ and $D_2 = \overline{A_1A_4} \cup \overline{A_2A_6} \cup \overline{A_3A_5}$. Then

$$C \cap D_1 \cap D_2 = C \cap D_1 = C \cap D_2 = \{A_1, \ldots, A_6\},$$

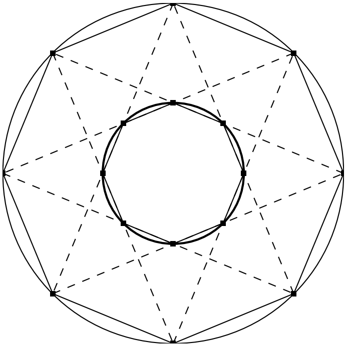
and $D_1 \cap D_2 = \{A_1, \ldots, A_6, P, Q, R\}$. We apply the corollary to see that $(D_1 \cap D_2) \setminus C = \{P, Q, R\}$ lie on a curve of degree $3 - 2 = 1$, i.e. a line. □

The case $n = 8$

We can extend these ideas to prove the following statement.

Theorem

If P is an 8-gon inscribed in a conic, then $T_{13}(P)$ is also inscribed in a conic.



The case $n = 8$

Proof.

Let A_i be the vertices of P and B_i the vertices of $T_{13}(P)$. Let C be the conic,

$$D_1 = \overline{A_1 A_4} \cup \overline{A_3 A_6} \cup \overline{A_5 A_8} \cup \overline{A_7 A_2},$$

$$D_2 = \overline{A_2 A_5} \cup \overline{A_4 A_7} \cup \overline{A_6 A_1} \cup \overline{A_8 A_3}.$$

We have

$$D_1 \cap D_2 = \{A_1, \dots, A_8, B_1, \dots, B_8\}$$

and

$$C \cap D_1 \cap D_2 = C \cap D_1 = C \cap D_2 = \{A_1, \dots, A_8\}.$$

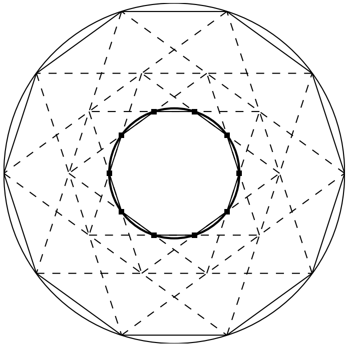
By our corollary we get that $(D_1 \cap D_2) \setminus C = \{B_1, \dots, B_8\}$ lie on a curve of degree $4 - 2 = 2$, i.e. a conic. □

The case $n = 10$

This case is a little more complicated.

Theorem

If P is a 10-gon inscribed in a conic, then $T_{1313}(P)$ is also inscribed in a conic.



The case $n = 10$

Proof.

Let A_i be the vertices of P and B_i the vertices of $T_{13}(P)$. Let C be the conic,

$$D_1 = \overline{A_1A_4} \cup \overline{A_3A_6} \cup \overline{A_5A_8} \cup \overline{A_7A_{10}} \cup \overline{A_9A_2},$$
$$D_2 = \overline{A_2A_5} \cup \overline{A_4A_7} \cup \overline{A_6A_1} \cup \overline{A_8A_1} \cup \overline{A_{10}A_3}.$$

We have

$$D_1 \cap D_2 = \{A_1, \ldots, A_{10}, B_1, \ldots, B_{10}, X_1, \ldots, X_5\}$$

where X_i are as follows:

$$X_1 = \overline{A_1A_4} \cap \overline{A_6A_9}, \quad X_2 = \overline{A_3A_6} \cap \overline{A_8A_1},$$
$$X_3 = \overline{A_5A_8} \cap \overline{A_{10}A_3}, \quad X_4 = \overline{A_7A_{10}} \cap \overline{A_2A_5},$$
$$X_5 = \overline{A_9A_2} \cap \overline{A_4A_7}.$$

□

The case $n = 10$

We also have

$$C \cap D_1 \cap D_2 = C \cap D_1 = C \cap D_2 = \{A_1, \dots, A_{10}\},$$

so by the corollary, $(D_1 \cap D_2) \setminus C = \{B_1, \dots, B_{10}, X_1, \dots, X_5\}$ lie on a curve C' of degree $5 - 2 = 3$.

Now let C_i be the vertices of $T_{1313}(P)$ and let

$$D_1' = \overline{B_1 B_4} \cup \overline{B_3 B_6} \cup \overline{B_5 B_8} \cup \overline{B_7 B_{10}} \cup \overline{B_9 B_2},$$

$$D_2' = \overline{B_2 B_5} \cup \overline{B_4 B_7} \cup \overline{B_6 B_1} \cup \overline{B_8 B_1} \cup \overline{B_{10} B_3}.$$

Define X_i' analogously to X_i (replacing A's by B's). Then we claim that X_i' lie on the cubic C'. We use a special case of the Cayley-Bacharach theorem, called Chasles' theorem.

Theorem

If two cubics in the projective plane intersect at nine points, then every cubic passing through any eight of them also passes through the ninth.

The case $n = 10$

In particular we can say that C' contains $B_1, B_7, B_3, B_6, B_2, B_8, X_2, X_4$ and thus must also contain $\overline{B_1 B_8} \cap \overline{B_3 B_6}$.

Applying the theorem again, C' contains $B_1, B_3, B_9, B_6, B_8, B_4, X_5, \overline{B_1 B_8} \cap \overline{B_3 B_6}$, so that $\overline{B_1 B_4} \cap \overline{B_6 B_9} = Y_1 \in C'$. Similarly, $Y_2, \ldots, Y_5 \in C'$.

Now we have that

$$D_1' \cap D_2' = \{B_1, \ldots, B_{10}, C_1, \ldots, C_{10}, Y_1, \ldots, Y_5\}$$

and

$$C' \cap D_1' \cap D_2' = C' \cap D_1' = C' \cap D_2' = \{B_1, \ldots, B_{10}, Y_1, \ldots, Y_5\},$$

so by the corollary $C_1, \ldots, C_{10}$ lie on a conic.

The case $n = 10$

We have one last thing to check: C' and D_1' must have no common components. We need to show that no line in the irreducible composition of D_1' is an irreducible component of C', so it's enough to check that $\overline{B_1 B_4}$ is not an irreducible component of C'.

If it were, then $C' = \overline{B_1 B_4} \cup \Gamma$ for some conic Γ. We have $X_4 = \overline{B_6 B_7} \cap \overline{B_1 B_2} \in C'$. Since $\overline{B_1 B_2} \cap \overline{B_1 B_4} = B_1$ and since $B_1 \notin \overline{B_6 B_7}$, we see $X_4 \notin \overline{B_1 B_4}$. Also, $B_6, B_7 \notin \overline{B_1 B_4}$, so $X_4, B_6, B_7 \in \Gamma$. But then we have three collinear points on a conic, which is impossible.

Further directions

We can apply the same ideas to the final case, $n = 12$, but we end up with points on a quartic curve, and it is unclear how to continue.

Another idea is to use the Cage Theorem for planar curves [Kat06]. We consider planar curves C and D of degree c and d that are both unions of lines, and we call $C \cup D$ a $(c \times d)$-cage. We color the lines, say C red and D blue.

A subset S of the nodes of a cage is called supra-quasi triangular (SQL) if (1) the number of nodes in S residing on a typical blue line ranges from d to $d - e + 2$, (2) the value d is achieved on two lines, (3) and each value from $d - 1$ to $d - e + 2$ is achieved exactly once.

Theorem (Cage)

If a projective plane curve of degree d passes through an SQL subset of nodes of a $(c \times d)$-cage, with $c \geq d$, then it passes through all nodes of the cage.

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